Category Theory 1

Categories and functors

This is to accompany the reading of 1–7 October and the lecture of 8 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip over any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Write down three examples of (a) categories, and (b) functors, that weren't given in lectures.

2. Show that functors preserve isomorphism. That is, prove that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor and $A, A' \in \mathcal{A}$ with $A \cong A'$, then $F(A) \cong F(A')$.

3. Two categories $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, written $\mathcal{A} \cong \mathcal{B}$, if they are isomorphic as objects of **CAT**.

   (a) Let G be a group, regarded as a one-object category. What is the opposite of G? Prove that G is isomorphic to G^{op}.

   (b) Find a monoid not isomorphic to its opposite.

4. Is there a functor $Z : \mathbf{Gp} \longrightarrow \mathbf{Gp}$ with the property that $Z(G)$ is the centre of G for all groups G?

5. *Sometimes we meet functors whose domain is a product $\mathcal{A} \times \mathcal{B}$ of categories. In this question we'll show that such a functor can be regarded as an interlocking pair of families of functors, one defined on $\mathcal{A}$ and one defined on $\mathcal{B}$. This is very like the situation with bilinear and linear maps.*

   Let $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ be a functor. For each $A \in \mathcal{A}$, there is an induced functor $F^A : \mathcal{B} \longrightarrow \mathcal{C}$ defined on objects $B \in \mathcal{B}$ by $F^A(B) = F(A, B)$ and on arrows g of $\mathcal{B}$ by $F^A(g) = F(1_A, g)$. Similarly, for each $B \in \mathcal{B}$ there is an induced functor $F_B : \mathcal{A} \longrightarrow \mathcal{C}$. Show that the families of functors $(F^A)_{A \in \mathcal{A}}$ and $(F_B)_{B \in \mathcal{B}}$ satisfy the following conditions:

   - if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $F^A(B) = F_B(A)$
   - if $f : A \longrightarrow A'$ in $\mathcal{A}$ and $g : B \longrightarrow B'$ in $\mathcal{B}$ then $F^{A'}(g) \circ F_B(f) = F_{B'}(f) \circ F^A(g)$.

   Then show that given families $(F^A)_{A \in \mathcal{A}}$ and $(F_B)_{B \in \mathcal{B}}$ of functors satisfying these two conditions, there is a unique functor $F : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ inducing them.

Category Theory 2

Natural transformations and equivalence

This is to accompany the reading of 11–17 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip over any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Write down three examples of natural transformations that aren't in the notes.

2. Prove that a natural transformation is a natural isomorphism if and only if each of its components is an isomorphism (Lemma 1.3.6).

3. *Linear algebra can be done equivalently with matrices or with linear maps...*

   Fix a field k. Let **Mat** be the category whose objects are the natural numbers and with

   $$\mathbf{Mat}(m, n) = \{n \times m \text{ matrices over } k\}.$$

   Prove that **Mat** is equivalent to **FDVect**, the category of finite-dimensional vector spaces over k. Does your equivalence involve a *canonical* functor from **Mat** to **FDVect**, or from **FDVect** to **Mat**?

   (Hints: (i) Part of the exercise is to work out what composition in the category **Mat** *is supposed to be; there's only one sensible possibility. (ii) It's easier if you use 1.3.12. (iii) The word 'canonical' means something like 'God-given' or 'definable without making any arbitrary choices'.)*

4. Let G be a group. For any $g \in G$ there is a unique homomorphism $\phi : \mathbb{Z} \longrightarrow G$ satisfying $\phi(1) = g$, so elements of G are essentially the same as homomorphisms $\mathbb{Z} \longrightarrow G$. These in turn are the same as functors $\mathbb{Z} \longrightarrow G$, where groups are regarded as one-object categories. Natural isomorphism therefore defines an equivalence relation on the elements of G. What is this equivalence relation, in group-theoretic terms?

   (First have a guess. For a general group G, what equivalence relations on G can you think of?)

5. A **permutation** on a set X is a bijection $X \longrightarrow X$. Let $\mathbf{Sym}(X)$ be the set of permutations on X. A **total order** on a set X is an order $\leq$ such that for all $x, y \in X$, either $x \leq y$ or $y \leq x$; so a total order on a finite set amounts to a way of placing its elements in sequence. Let $\mathbf{Ord}(X)$ be the set of total orders on X.

   Let $\mathcal{B}$ be the category of finite sets and bijections.

   (a) Give a definition of **Sym** on morphisms of $\mathcal{B}$ so that **Sym** becomes a functor $\mathcal{B} \longrightarrow \mathbf{Set}$. Do the same for **Ord**. Both your definitions should be canonical (no arbitrary choices).

   (b) Show that there is no natural transformation $\mathbf{Sym} \longrightarrow \mathbf{Ord}$. *(Hint: consider the identity permutation.)*

   (c) If X is an n-element set, how many elements do the sets $\mathbf{Sym}(X)$ and $\mathbf{Ord}(X)$ have?

   Conclude that $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$ for all $X \in \mathcal{B}$, but not naturally in X.

Category Theory 3
Adjoints

This is to accompany the reading of 17–24 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Write down two examples of (a) adjunctions, (b) initial objects, and (c) terminal objects, that aren't in the notes.

2. What can you say about adjunctions between discrete categories?

3. What is an **adjunction**? Show that left adjoints preserve initial objects, that is, if $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}}_{\bot} \mathcal{B}$ and I is an initial object of $\mathcal{A}$, then $F(I)$ is an initial object of $\mathcal{B}$. Dually, show that right adjoints preserve terminal objects.

   (Later we'll see this as part of a bigger picture: right adjoints preserve limits and left adjoints preserve colimits.)

4.(a) Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}}_{\bot} \mathcal{B}$ be an adjunction. Define the **unit** η and **counit** ε of the adjunction. Prove the triangle identities, $(\varepsilon F) \circ (F\eta) = 1_F$ and $(G\varepsilon) \circ (\eta G) = 1_G$.

   (b) Prove that given functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ and natural transformations $\eta : 1 \longrightarrow GF$, $\varepsilon : FG \longrightarrow 1$ satisfying the triangle identities, there is a unique adjunction between F and G with η as its unit and ε as its counit.

5. Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}}_{\bot} \mathcal{B}$ be an adjunction with unit η and counit ε. Let $\mathbf{Fix}(GF)$ be the full subcategory of $\mathcal{A}$ whose objects are those $A \in \mathcal{A}$ for which η_A is an isomorphism, and dually $\mathbf{Fix}(FG) \subseteq \mathcal{B}$. Prove that the adjunction $(F, G, \eta, \varepsilon)$ restricts to an equivalence $(F', G', \eta', \varepsilon')$ between $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$.

   In this way, any adjunction restricts to an equivalence between full subcategories. Take some examples of adjunctions and work out what this equivalence is.

Category Theory 4
Adjoints and sets

This is to accompany the reading of 24–31 October. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $G : \mathcal{B} \longrightarrow \mathcal{A}$ be a functor.

   (a) For $A \in \mathcal{A}$, define the comma category $(A \Rightarrow G)$.

   (b) Suppose that G has a left adjoint F, and let η be the unit of the adjunction. Show that η_A is an initial object of $(A \Rightarrow G)$, for each $A \in \mathcal{A}$.

   (c) Conversely, suppose that for each $A \in \mathcal{A}$, the category $(A \Rightarrow G)$ has an initial object. Show that G has a left adjoint.

2. State the dual of Corollary 2.3.6. What would you do if someone asked you to prove your dual statement? *(Duality is discussed in Remark 2.1.7.)*

3. The **diagonal functor** $\Delta : \mathbf{Set} \longrightarrow \mathbf{Set}^2$ is defined by $\Delta(A) = (A, A)$ for all sets A. Exhibit left and right adjoints to Δ.

4. Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor sending a small category to its set of objects. Exhibit a chain of adjoints
$$C \dashv D \dashv O \dashv I.$$

Category Theory 5

Representables

This is to accompany the reading of 31 October–7 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

Some questions on these sheets require knowledge of other areas of mathematics; skip any that you haven't the background for. That aside, I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $\mathcal{A}$ be a locally small category. What does it mean for a presheaf X on $\mathcal{A}$ to be **representable**? What is a **representation** of X?

2. Write down five examples of representable functors. *(It's possible to answer this with almost no inventiveness at all: just look at the definition of representability.)*

3. Let $\mathcal{A}$ be a locally small category and $A, B \in \mathcal{A}$. Show that if $H_A \cong H_B$ then $A \cong B$.

4. *One way to understand the Yoneda Lemma is to think about special cases. Here we think about one-object categories.*

   Let M be a monoid. The underlying set of M can be given a right M-action by multiplication: $x \cdot m = xm$ for all $x, m \in M$. This M-set is called the **right regular representation** of M. I will write it as $\underline{M}$.

   (a) When M is regarded as a one-object category, functors $M^{\mathrm{op}} \longrightarrow \mathbf{Set}$ correspond to right M-sets. Show that the M-set corresponding to the unique representable functor $M^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is the right regular representation.

   (b) Now let X be any right M-set. Show that for each $x \in X$, there is a unique map $\alpha : \underline{M} \longrightarrow X$ of right M-sets such that $\alpha(1) = x$. *(See 1.3.3(b) for the definition of a map of M-sets; those are left M-sets but you can dualize.)* Deduce that there is a bijection between {maps $\underline{M} \longrightarrow X$ of right M-sets} and X.

   (c) Deduce the Yoneda Lemma for one-object categories.

5. *Here we consider natural transformations between functors whose domain is a product category $\mathcal{A} \times \mathcal{B}$. Your task is to show that naturality in two variables simultaneously is equivalent to naturality in each variable separately.*

   Take functors $F, G : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$. For each $A \in \mathcal{A}$ there are functors $F^A, G^A : \mathcal{B} \longrightarrow \mathcal{C}$, as in Sheet 1, q.5; similarly, for each $B \in \mathcal{B}$, there are functors $F_B, G_B : \mathcal{A} \longrightarrow \mathcal{C}$.

   Let $(\alpha_{A,B} : F(A, B) \longrightarrow G(A, B))_{A \in \mathcal{A}, B \in \mathcal{B}}$ be any family of maps. Show that this family is a natural transformation $F \longrightarrow G$ if and only if

   - for each $A \in \mathcal{A}$, the family $(\alpha_{A,B} : F^A(B) \longrightarrow G^A(B))_{B \in \mathcal{B}}$ is a natural transformation $F^A \longrightarrow G^A$, and
   - for each $B \in \mathcal{B}$, the family $(\alpha_{A,B} : F_B(A) \longrightarrow G_B(A))_{A \in \mathcal{A}}$ is a natural transformation $F_B \longrightarrow G_B$.

Category Theory 6
The Yoneda Lemma

This is to accompany the reading of 7 November–14 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. State and prove the Yoneda Lemma.

2. Let $\mathcal{A}$ be a locally small category. Define the **Yoneda embedding** $H_\bullet : \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Prove each of the following directly (without using the Yoneda Lemma):

   (a) $H_\bullet$ is faithful

   (b) $H_\bullet$ is full

   (c) if $X : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $A \in \mathcal{A}$, and $X(A)$ has an element u that is 'universal' in a sense that you should make precise, then $X \cong H_A$.

3. Prove Lemma 3.3.8.

4. Let $\mathcal{B}$ be a category and $J : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. There is an induced functor

$$J \circ - : [\mathcal{B}, \mathcal{C}] \longrightarrow [\mathcal{B}, \mathcal{D}]$$

   defined by composition with J. *(If you can't see how J is defined on maps, look back at Remark 1.3.14.)*

   (a) Show that if J is full and faithful then so is $J \circ -$. *(Typical category theory question. It's straightforward in the sense that nothing sneaky's involved: you just follow your nose. On the other hand, it may take you a while to get oriented. Remain calm.)*

   (b) Deduce that if J is full and faithful and $G, G' : \mathcal{B} \longrightarrow \mathcal{C}$ with $J \circ G \cong J \circ G'$ then $G \cong G'$.

   (c) Now deduce that right adjoints are unique: if $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G, G' : \mathcal{B} \longrightarrow \mathcal{A}$ with $F \dashv G$ and $F \dashv G'$ then $G \cong G'$. *(Hint: the Yoneda embedding is full and faithful.)*

Category Theory 7
Limits

This is to accompany the reading of 14–21 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Define the term **limit**. In what sense are limits unique? Prove your uniqueness statement.

2. *Limit is a process that takes a diagram of shape $\mathbb{I}$ in a category $\mathcal{A}$, and produces from it a new object of $\mathcal{A}$. Later we'll see that this process is functorial. Here we show this in the special case of binary products.*

   Let $\mathcal{A}$ be a category with binary products. Choose for each pair (X, Y) of objects a product cone
   $$X \xleftarrow{p_1^{X,Y}} X \times Y \xrightarrow{p_2^{X,Y}} Y.$$
   Show that once this choice is made, we have a canonical functor $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ defined on objects by $(X, Y) \longmapsto X \times Y$.

3. Take a commutative diagram
   $$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$
   in some category. Suppose that the right-hand square is a pullback. Show that the left-hand square is a pullback if and only if the outer rectangle is a pullback.

4. Let $E \xrightarrow{i} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be an equalizer (in some category). Is
   $$\begin{array}{ccc} E & \xrightarrow{i} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$
   necessarily a pullback? Give a proof or a counterexample.

5. A map $m : A \longrightarrow B$ in a category is **regular monic** if there exist an object C and maps $B \rightrightarrows C$ of which m is an equalizer. It is **split monic** if there exists a map $e : B \longrightarrow A$ such that $em = 1_A$.
   (a) Show that split monic $\Rightarrow$ regular monic $\Rightarrow$ monic.
   (b) In **Ab**, show that all monics are regular but not all monics are split. *(Hint for the first part: equalizers in* **Ab** *are calculated as in* $\mathbf{Vect}_k$*.)*
   (c) In **Top**, describe the regular monics and find a monic that is not regular.

Category Theory 8

Limits and colimits

This is to accompany the reading of 21–28 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $\mathbb{I}$ be a small category and $D : \mathbb{I} \longrightarrow \mathbf{Set}$ a diagram of shape $\mathbb{I}$ in **Set**. Describe explicitly a limit cone and a colimit cocone for D.

2. What does it mean for a functor to **preserve**, **reflect** or **strictly create** limits? Show that if $F : \mathcal{A} \longrightarrow \mathcal{B}$ strictly creates limits and $\mathcal{B}$ has all limits, then $\mathcal{A}$ has all limits and F preserves them.

3. Let $\mathcal{A}$ be a category with binary products. Show that

$$\mathcal{A}(A, B \times C) \cong \mathcal{A}(A, B) \times \mathcal{A}(A, C)$$

naturally in $A, B, C \in \mathcal{A}$.

(I'm assuming implicitly that we've chosen for each B and C a product cone on (B, C). By Sheet 7, q.2, the assignment $(B, C) \longmapsto B \times C$ is then functorial—which it would have to be in order for the word 'naturally' in the question to make sense.)

4. Let $\mathbb{I}$ be a small category. Show that a category $\mathcal{A}$ has all limits of shape $\mathbb{I}$ if and only if the diagonal functor $\Delta : \mathcal{A} \longrightarrow [\mathbb{I}, \mathcal{A}]$ has a right adjoint.

5. Recall the definitions of regular monic and split monic from Sheet 7, q.5.

   (a) Give an example, with proof, of a map in a category that is monic and epic but not an isomorphism.

   (b) Prove that in any category, a map is an isomorphism if and only if it is both monic and regular epic.

   (c) Assuming that our category of sets satisfies the Axiom of Choice (page 40 of the notes), show that

$$\text{epic} \iff \text{regular epic} \iff \text{split epic}$$

   in **Set**.

(You can say that a category $\mathcal{A}$ 'satisfies the Axiom of Choice' if all epics in $\mathcal{A}$ are split. For example, the Axiom of Choice is not *satisfied in* **Top** *or in* **Gp**.*)*

Category Theory 9

Limits and colimits of presheaves

This is to accompany the reading of 28 November–5 December. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Let $\mathbb{A}$ be a small category.

   (a) What does it mean to say that limits and colimits are computed pointwise in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$? Prove that this is so.

   (b) Describe explicitly the monics and epics in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$. (Now see if you can do this without the aid of (a).)

2. Let $\mathbb{A}$ be a small category.

   (a) Show that for each $A \in \mathbb{A}$, the representable functor $H^A : \mathbb{A} \longrightarrow \mathbf{Set}$ preserves limits.

   (b) Show that the Yoneda embedding $H_\bullet : \mathbb{A} \longrightarrow [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves limits.

3. Let $\mathbb{A}$ be a small category and $A, B \in \mathbb{A}$. Show that the sum $H_A + H_B$ in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ is never representable.

   (Warning 5.1.13 might give a clue. You might also want to use the description of representability in terms of universal elements, though you don't need to.)

4. Let X be a presheaf on a small category. Show that X is representable if and only if its category of elements $\mathbb{E}(X)$ has a terminal object.

   Since a terminal object is a limit of the empty diagram, this means that the concept of representability can be derived from the concept of limit. Since a terminal object of a category $\mathcal{E}$ is a right adjoint to the unique functor $\mathcal{E} \longrightarrow \mathbf{1}$, representability can also be derived from the concept of adjoint.

5. Let $\mathcal{A}$ be a category and $A \in \mathcal{A}$. A **subobject** of A is an isomorphism class of monics into A. More precisely, let $\mathbf{Monic}(A)$ be the category whose objects are the monics with codomain A and whose maps are commutative triangles; this is a full subcategory of the slice category $\mathcal{A}/A$ (Example 2.3.3(a)). Then a subobject of A is an isomorphism class of objects of $\mathbf{Monic}(A)$.

   (a) Let $X \xrightarrow{m} A$ and $X' \xrightarrow{m'} A$ be monics in $\mathbf{Set}$. Show that m and m' are isomorphic in $\mathbf{Monic}(A)$ if and only if they have the same image. Deduce that subobjects of A correspond one-to-one with subsets of A.

   (b) Part (a) says that in $\mathbf{Set}$, subobjects are subsets. What are subobjects in $\mathbf{Gp}$, $\mathbf{Ring}$ and $\mathbf{Vect}_k$? How about in $\mathbf{Top}$? *(Careful!)*

Category Theory 10

Interaction of (co)limits with adjunctions

This is to accompany the final batch of reading, beginning on 5 December. In the week of 7–11 January, there will be a lecture on this material and a tutorial on this sheet. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. Consider the following three conditions on a functor U from a locally small category $\mathcal{A}$ to **Set**:

   A. U has a left adjoint **R**. U is representable **L**. U preserves limits.

   (a) Show that $\mathbf{A} \Rightarrow \mathbf{R} \Rightarrow \mathbf{L}$.
   (b) Show that if $\mathcal{A}$ has sums then $\mathbf{R} \Rightarrow \mathbf{A}$.

   (If $\mathcal{A}$ satisfies the hypotheses of the Special Adjoint Functor Theorem then $\mathbf{L} \Rightarrow \mathbf{A}$ *and the three conditions are equivalent.)*

2. (a) Prove that left adjoints preserve colimits and right adjoints preserve limits.
   (b) Prove that the forgetful functor $U : \mathbf{Gp} \longrightarrow \mathbf{Set}$ has no right adjoint.
   (c) Prove that the chain of adjunctions $C \dashv D \dashv O \dashv I$ in Sheet 4, q.4 extends no further in either direction.

3. (a) Show that every presheaf on a small category is a colimit of representable presheaves.
   (b) What does it mean for a category to be **cartesian closed**? Show that for any small category $\mathbb{A}$, the presheaf category $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. (You may assume that limits and colimits in presheaf categories exist and are computed pointwise.)

4. (a) Let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A' & \xrightarrow[f]{} & A \end{array}$$

   be a pullback square in some category. Show that if m is monic then so is m'. *(We already know that this holds in the category of sets: Example 4.1.16.)*

   A category $\mathcal{A}$ is **well-powered** if for each $A \in \mathcal{A}$, the class of subobjects of A is small—that is, a set. All of our usual examples of categories are well-powered. Let $\mathcal{A}$ be a well-powered category with pullbacks, and write $\mathrm{Sub}(A)$ for the set of subobjects of an object $A \in \mathcal{A}$.

   (b) Deduce from (a) that any map $A' \xrightarrow{f} A$ in $\mathcal{A}$ induces a map $\mathrm{Sub}(A) \xrightarrow{\mathrm{Sub}(f)} \mathrm{Sub}(A')$.
   (c) Show that this determines a functor $\mathrm{Sub} : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. *(Hint: Sheet 7, q.3.)*
   (d) For some categories $\mathcal{A}$, Sub is representable. A **subobject classifier** for $\mathcal{A}$ is an object $\Omega \in \mathcal{A}$ such that $\mathrm{Sub} \cong H_\Omega$. Prove that 2 is a subobject classifier for **Set**.

Category Theory

Hints on the problem sheets

I've written varying amounts about each question. Sometimes it's just a quick hint and sometimes it's something more detailed—but almost none of my answers are up to the level of detail expected in an exam.

General hint Before you look here for a hint,

make sure you understand the question in full.

In category theory, maybe more than in most subjects, you really have to completely understand every piece of terminology used in the question before trying to answer it. If you don't, you're extremely unlikely to produce a correct answer. But once you do, you may well find the answer a pushover. The purpose of these questions is to deepen and test your understanding, not to exercise your problem-solving skills. It's not like number theory or combinatorics, where there are many questions that can be stated in simple terms but are very hard to answer.

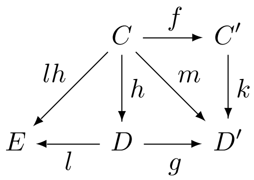
So, the questions are often harder than the answers! This is particularly true of the questions on the earlier sheets.

Sheet 1: Categories and functors

1. For everyday examples of categories and functors, browse library or web. Or you can make up examples in the following manner. There's a category

$$\mathcal{A} = (A \xrightarrow{p} B)$$

—that is, $\mathcal{A}$ has two objects, A and B, and just one non-identity map, $p : A \longrightarrow B$. (No need to say what composition is, as that's uniquely determined.) Or (random example) there's a category $\mathcal{B}$ with objects and maps



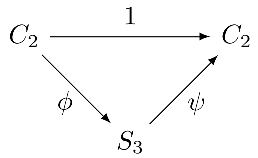
where $gh = kf = m$ and I've omitted identity maps. There's a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ defined by $F(A) = C$, $F(B) = C'$, and $F(p) = f$.

2. See 'General hint' above.

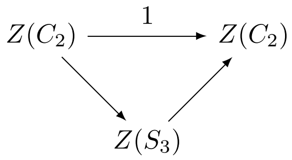
3. (a) Same set but multiplication reversed: $(a, b) \longmapsto b \cdot a$. Isomorphism $G \longrightarrow G^{\text{op}}$ provided by $g \longmapsto g^{-1}$.

   (b) Let M be the monoid of maps $2 \longrightarrow 2$ where 2 is a two-element set and multiplication is composition. Then the statement $\exists m \in M : \forall x \in M, mx = m$ is true, but becomes false when M is replaced by M^{op}. So $M \not\cong M^{\text{op}}$.

4. No. Main point: a homomorphism $\phi : G \longrightarrow H$ doesn't restrict to a map $Z(G) \longrightarrow Z(H)$ (e.g. take an injection $\phi : C_2 \hookrightarrow S_3$). So the *obvious* way of defining Z on maps fails. In fact there's *no* way to do it: for if there were, the commutative diagram



(where ψ is the quotient map for $A_3 \trianglelefteq S_3$) would become a commutative diagram

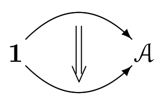


which is impossible as $Z(C_2) = C_2$ and $Z(S_3) \cong 1$.

5. Easy once you fully understand the question. Write out the definition of $\mathcal{A} \times \mathcal{B}$ *in full*: what the objects, maps, composition and identities are. Write down *in full* what a functor $\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ is. Then try it.

Sheet 2: Natural transformations and equivalence

1. For examples that occur mathematical practice, browse library or web. Can also make up examples as in hints to Sheet 1, q.1. E.g. if $\mathbf{1}$ is the category with one object and one map (the identity) then a functor from $\mathbf{1}$ to a category $\mathcal{A}$ is just an object of $\mathcal{A}$, and a natural transformation



between two such functors is a map in $\mathcal{A}$ between the corresponding two objects. Or, take the categories $\mathcal{A}$ and $\mathcal{B}$ defined in the hints to Sheet 1, q.1: then there is a functor F as defined there, another functor G defined by $G(p) = g$, and a natural transformation $\alpha : F \longrightarrow G$ given by $\alpha_A = h$ and $\alpha_B = k$.

2. See 'General hint' above.

3. Define $F : \mathbf{Mat} \longrightarrow \mathbf{FDVect}$ as follows: $F(n) = k^n$, and if $M \in \mathbf{Mat}(m, n)$ then $F(M)$ is the linear map $k^m \longrightarrow k^n$ corresponding to the matrix M (with respect to the standard bases). Show functorial. Show full and faithful and essentially surjective on objects. Invoke 1.3.12.

   This functor F is canonical, but there's no canonical functor $G : \mathbf{FDVect} \longrightarrow \mathbf{Mat}$ satisfying $FG \cong 1$ and $GF \cong 1$: for such a G must send every finite-dimensional vector space V to $\dim V$ (fine), but to specify G on maps, you'd have to choose a basis for every finite-dimensional vector space, which can't be done in a canonical way.

4. Conjugacy.

5.(a) Let $f : X \longrightarrow Y$ be a map in $\mathcal{B}$. Then $\mathbf{Sym}(f) : \mathbf{Sym}(X) \longrightarrow \mathbf{Sym}(Y)$ is defined by $\sigma \longmapsto f\sigma f^{-1}$. Also $\mathbf{Ord}(f) : \mathbf{Ord}(X) \longrightarrow \mathbf{Ord}(Y)$ is defined by $\leq \longmapsto \leq'$ where $y_1 \leq' y_2 \iff f^{-1}(y_1) \leq f^{-1}(y_2)$. Check functoriality.

(b) Take $\alpha : \mathbf{Sym} \longrightarrow \mathbf{Ord}$. Draw naturality square for α with respect to the map $f : 2 \longrightarrow 2$ in $\mathcal{B}$ where 2 is a two-element set and f interchanges its elements. Work out what its commutativity says when you take the identity permutation $1 \in \mathbf{Sym}(2)$: get contradiction.

Sheet 3: Adjoints

1. Same comments as for Sheet 1, q.1 and Sheet 2, q.1.

2. They are just bijections between sets (or strictly speaking, classes): if $F \dashv G$ is an adjunction between discrete categories $\mathcal{A}$ and $\mathcal{B}$ then F is an isomorphism and $G = F^{-1}$.

3. For all B, the set $\mathcal{B}(F(I), B) \cong \mathcal{A}(I, G(B))$ has one element. And dually.

4. Bookwork.

5. The substantial parts are (i) understanding the concepts behind the question, and (ii) observing that if η_A is an isomorphism then so is $\varepsilon_{F(A)}$ (by a triangle identity) and dually.

   The equivalence you restrict to can be completely trivial, e.g. the adjunction $\mathbf{Vect}_k \rightleftarrows \mathbf{Set}$ becomes the equivalence $\emptyset \rightleftarrows \emptyset$ (where $\emptyset$ is the empty category). Slightly less trivial: $\mathbf{Top} \underset{D}{\overset{U}{\rightleftarrows}} \mathbf{Set}$ gives the equivalence (discrete spaces) $\simeq \mathbf{Set}$.

Sheet 4: Adjoints and sets

1. Bookwork.

2. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. Then F has a right adjoint if and only if for each $B \in \mathcal{B}$, the category $(F \Rightarrow B)$ has a terminal object.

   Proof: can just say 'by duality'.

3. Left: $(A, B) \longmapsto A + B$. Right: $(A, B) \longmapsto A \times B$.

4. I can think of three general strategies for finding adjoints. You can use them to find D, I and C respectively.

   Guess it We're given $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ and want to know what its adjoints are. Have a guess: what functors $\mathbf{Set} \longrightarrow \mathbf{Cat}$ do we already know? In other words, what methods do we know for constructing a category out of a set? One is the discrete category construction (1.3.3(a)), which defines a functor $D : \mathbf{Set} \longrightarrow \mathbf{Cat}$. Check that this is left adjoint to O.

Probe it We're told that O has a right adjoint I. We can try to figure out what it must be by using adjointness. Given a set S, an object of $I(S)$ is a functor $\mathbf{1} \longrightarrow I(S)$, which is a function $O(\mathbf{1}) \longrightarrow S$, which is an element of S. So the object-set of $I(S)$ is S. An arrow in $I(S)$ is a functor $\mathbf{2} \longrightarrow I(S)$ (where $\mathbf{2}$ is the category $\mathcal{A}$ in the hint to Sheet 1, q.1), which is a function $O(\mathbf{2}) \longrightarrow S$, which is a pair of elements of S. So the arrow-set of $I(S)$ is $S \times S$. You could carry on with this method to figure out what domain, codomain, composition and identities are in $I(S)$, but perhaps you can now make the leap and guess it: $I(S)$ is the category whose objects are the elements of S, where for each $A, B \in S$ there is exactly one map $A \longrightarrow B$, and where composition and identities are defined in the only possible way. It's called the **indiscrete category** on S.

Stare at it We'll use this to find C. Let $\mathbb{A}$ be a category and S a set. A functor $F : \mathbb{A} \longrightarrow D(S)$ is supposed to be the same thing as a function $C(\mathbb{A}) \longrightarrow S$, whatever C is. Well, what *is* a functor $F : \mathbb{A} \longrightarrow D(S)$? It's a way of assigning to every object $A \in \mathbb{A}$ an element $F(A)$ of S, with the property that for every map $A \xrightarrow{f} B$ in $\mathbb{A}$ we have $F(A) = F(B)$. In other words (aha!), it's a function $O(\mathbb{A})/\sim \longrightarrow S$ where $\sim$ is the equivalence relation on $O(\mathbb{A})$ generated by $A \sim B$ whenever there's a map $A \longrightarrow B$. So $C(\mathbb{A}) = O(\mathbb{A})/\sim$. This is called the set of **connected-components** of $\mathbb{A}$.

Sheet 5: Representables

1. Bookwork.

2. The non-inventive answer: by definition, theere's one representable for every pair $(\mathcal{A}, A)$ where $\mathcal{A}$ is a category and $A \in \mathcal{A}$, namely H^A. So to give five examples of representable functors, you can just write down five examples of objects of categories!

   For more interesting answers, browse library/web.

3. Take isomorphism $\alpha : H_A \longrightarrow H_B$. We have to define maps $A \underset{g}{\overset{f}{\rightleftarrows}} B$ and prove $gf = 1_A$ and $fg = 1_B$. Define $f = \alpha_A(1_A)$ and $g = \alpha_B(1_B)$. (What else could we possibly do?) Get $gf = 1_A$ and $fg = 1_B$ from naturality of α.

4. (a) Pushover once you fully understand the question: e.g. make sure you fully understand how monoids are one-object categories and M-sets are functors $M \longrightarrow$ **Set**. If it helps, use a different letter ($\mathcal{M}$, say) for the one-object category corresponding to the monoid M.

   (b) The unique map α is $m \longmapsto xm$. The bijection is $\phi \longmapsto \phi(1)$. (Moral: unique existence statements can be rephrased as saying that some function is a bijection.)

   (c) This is just (b) rephrased.

   (Well, the statement of the Yoneda Lemma also includes naturality in X and in the object (usually called 'A'). We haven't proved this part, although we know that our bijection is natural in the sense of being canonically defined—no random choices involved.)

5. Same kind of comments as for Sheet 1, q.5.

Sheet 6: The Yoneda Lemma

1. Bookwork.
2. Definition of Yoneda embedding: bookwork.
   (a) If $f : A \longrightarrow B$ is a map in $\mathcal{A}$ then $f = H_f(1_A)$.
   (b) Given $\alpha : H_A \longrightarrow H_B$, define $f = \alpha_A(1_A)$; show $H_f = \alpha$.
   (c) Definition of universality: see 3.3.2. Isomorphism $\alpha : H_A \longrightarrow X$ given by $\alpha_B(g) = (Xg)(x)$.

3.(a) If $J(f)$ is an isomorphism then by fullness, there exists a map $f' : C' \longrightarrow C$ such that $J(f') = J(f)^{-1}$; then check that $J(f'f) = 1$, which by faithfulness implies $f'f = 1$, and similarly $ff' = 1$.
   (b) Use (a).
   (c) Follows from (b).

4.(a) As the hint on the problem sheet suggests, it's easy once you understand the question. If you're having trouble, try writing out in full the definition of the functor $J \circ -$ (i.e. what it does to objects *and* to maps).
   (b) Follows from (a) and q.3.
   (c) Take $\mathcal{C} = \mathcal{A}$ and $J = H_\bullet$ in (b).

Sheet 7: Limits

1. Definition of limit: bookwork.

   Uniqueness: there are at least three statements you might make. Let $D : \mathbb{I} \longrightarrow \mathcal{A}$ be a diagram and take limit cones $(L \xrightarrow{p_I} D(I))_{I \in \mathbb{I}}$ and $(L' \xrightarrow{p'_I} D(I))_{I \in \mathbb{I}}$.

   Weakest $L \cong L'$.

   Stronger There is a unique isomorphism $j : L \longrightarrow L'$ such that $p'_I \circ j = p_I$ for all I.

   Strongest There is a unique map $j : L \longrightarrow L'$ such that $p'_I \circ j = p_I$ for all I, and j is an isomorphism.

   I'll prove the strongest. First half of statement holds because $(L' \xrightarrow{p'_I} D(I))_{I \in \mathbb{I}}$ is a limit cone. Similarly, have unique map $j' : L' \longrightarrow L$ such that $p_I \circ j' = p'_I$ for all I. Then $j'j : L \longrightarrow L$ satisfies $p_I \circ j'j = p_I$ for all I, and 1_L satisfies $p_I \circ 1_L = p_I$ for all I; but $(L \xrightarrow{p_I} D(I))_{I \in \mathbb{I}}$ is a limit cone, so $j'j = 1_L$. Similarly, $jj' = 1_L$. So j is an isomorphism.

2. We define a functor $F : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ given on objects by $F(X, Y) = X \times Y$.

   Given a map $(X, Y) \xrightarrow{(f,g)} (X', Y')$ in $\mathcal{A} \times \mathcal{A}$, there is a unique map $h : X \times Y \longrightarrow X' \times Y'$ such that

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_1^{X,Y}} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ X' \times Y' & \xrightarrow[p_1^{X',Y'}]{} & X' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X \times Y & \xrightarrow{p_2^{X,Y}} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X' \times Y' & \xrightarrow[p_2^{X',Y'}]{} & Y' \end{array}$$

commute, since $(X' \xleftarrow{p_1^{X',Y'}} X' \times Y' \xrightarrow{p_2^{X',Y'}} Y')$ is a product cone. Define $F(f,g) = h$. Check functoriality. To justify the word 'canonical', observe that in this answer we've done nothing random (unlike the question-setter, who randomly chose a product cone on every pair of objects).

3. Do 'if' and 'only if' separately. The only thing you've got to work with is the definition of pullback, and there's only one way to proceed.

4. No. E.g. if $f = g$ then i is an isomorphism, but then the square is a pullback if and only if f is monic (see 4.1.31). So we get a counterexample from any non-monic map. For instance, take f and g both to be the unique map $2 \longrightarrow 1$ in **Set**.

5.(a) If m is split monic with $em = 1$ then m is equalizer of $B \overset{me}{\underset{1}{\rightrightarrows}} B$. If m is regular monic then the uniqueness part of the definition of equalizer implies that m is monic.

(b) Any monic $m : A \longrightarrow B$ in **Ab** is the equalizer of $B \overset{q}{\underset{0}{\rightrightarrows}} B/\text{im}(m)$ where q is the quotient map (much as in 4.1.15(c)). The map $m : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $m(x) = 2x$ is injective, therefore monic. It is not split monic: if $em = 1$ then $e(2) = 1$, so $2e(1) = 1$, and there is no integer x satisfying $2x = 1$.

(c) In **Top**, a map is monic iff injective (arguing as in 4.1.30(a)). A map $m : A \longrightarrow B$ is regular monic iff the induced map $A \longrightarrow m(A)$ is a homeomorphism. (So up to isomorphism, the regular monics are the inclusions of subspaces.) In particular, a bijection m is regular monic if and only if it is a homeomorphism, so we get an example by writing down any example of a continuous bijection that is not a homeomorphism. For instance, let A be $\mathbb{R}$ with the discrete topology, let B be $\mathbb{R}$ with the usual topology, and let m be the map that is the identity on underlying sets. Or let $A = [0, 1)$, let B be the circle, thought of as consisting of the complex numbers of modulus 1, and put $m(t) = e^{2\pi i t}$.

Sheet 8: Limits and colimits

1. Bookwork.

2. Definitions: bookwork. Second part is straight manipulation of definitions.

3. Choose a product cone on every pair (B, C), with notation as in Sheet 7, q.2. For each A, B, C, define a function

$$\begin{array}{rccc} \alpha_{A,B,C} : & \mathcal{A}(A, B \times C) & \longrightarrow & \mathcal{A}(A, B) \times \mathcal{A}(A, C) \\ & \overline{q} & \longmapsto & (p_1^{B,C} \circ \overline{q},\ p_2^{B,C} \circ \overline{q}), \end{array}$$

which is bijective by definition of limit. Prove α natural.

4. 'Only if' is bookwork. For 'if', write R for the right adjoint of Δ. Let $D \in [\mathbb{I}, \mathcal{A}]$. Then $[\mathbb{I}, \mathcal{A}](\Delta A, D) \cong \mathcal{A}(A, R(D))$ naturally in $A \in \mathcal{A}$. Applying 4.4.2, conclude that $R(D)$ is a limit of D.

5.(a) Simplest of many possibilities: take the unique non-identity map in the category $\mathcal{A} = (\bullet \longrightarrow \bullet)$.

(b) Follows from observation that $X \overset{p}{\underset{p}{\rightrightarrows}} Y \xrightarrow{q} Z$ is a coequalizer if and only if q is an isomorphism.

(c) Axiom of Choice (page 40) says exactly that epic $\Rightarrow$ split epic in **Set**. Then use dual of Sheet 7, q.5.

Sheet 9: Limits and colimits of presheaves

1.(a) The meaning of 'computed pointwise' is the statement of Theorem 5.1.5 (with $\mathbb{A}$ changed to $\mathbb{A}^{\mathrm{op}}$ and $\mathbb{S}$ to **Set**).

(b) Applying Lemma 4.1.31, a map α in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ is monic iff a certain square involving α is a pullback, iff for each $A \in \mathbb{A}$ the analogous square involving α_A is a pullback (since pullbacks are computed pointwise), iff for each $A \in \mathbb{A}$ the map α_A is monic. The monics in **Set** are the injections, so α is monic iff each α_A is injective. Similarly, the epics are the pointwise surjections.

Without using (a), can still figure out what the monics are: do a direct proof by considering maps out of representables. But I know of no way of proving the result on epics without (a).

2. Bookwork.

3. Something stronger is true: every representable H_C is **connected**, meaning that whenever $H_C \cong X + Y$ for presheaves X and Y, then $X \cong 0$ or $Y \cong 0$. (Here $0 = \Delta\emptyset$ is the initial presheaf.) This implies the result in the question because $H_A \not\cong 0$ (since we have $1_A \in H_A(A)$) and similarly $H_B \not\cong 0$.

(Actually, connectedness also includes the condition of not being isomorphic to 0. This is very like the condition that 1 is not a prime number.)

To prove that H_C is connected, suppose $H_C = X + Y$. Then have universal element $u \in (X+Y)(C) \cong X(C)+Y(C)$. Viewing $X(C)$ and $Y(C)$ as subsets of $(X+Y)(C)$, either $u \in X(C)$ or $u \in Y(C)$. If $u \in X(C)$ then $((X+Y)(f))(u) \in X(D)$ for all maps $D \xrightarrow{f} C$, which implies (by definition of universality) that $Y(D) = \emptyset$ for all D; hence $Y \cong 0$. Similarly, if $u \in Y(C)$ then $X \cong 0$.

4. Follows immediately from 3.3.2 and definition of $\mathbb{E}(X)$.

5.(a) If you're having trouble with 'only if', make sure you understand the definition of $\mathbf{Monic}(A)$; perhaps 2.3.3(a) will help. For 'if', write I for the common image of m and m'; then since monic = injective in **Set**, there is a bijection $j : X \longrightarrow I$ defined by $j(x) = m(x)$, and similarly $j' : X' \longrightarrow I$; show $(j')^{-1} \circ j$ is an isomorphism from m to m'.

(b) Subgroups, subrings, vector subspaces. In **Top**, a subobject is a subset equipped with a topology containing the subspace topology. (If you'd prefer the answer to be 'subspaces', take *regular* subobjects instead: equivalence classes of regular monics. See Sheet 7, q.5(c).)

Sheet 10: Interaction of (co)limits with adjunctions

1.(a) Bookwork.

(b) Given $A \in \mathcal{A}$, have to find left adjoint to $H^A : \mathcal{A} \longrightarrow \mathbf{Set}$. For $S \in \mathbf{Set}$ and $B \in \mathcal{A}$, a map $S \longrightarrow H^A(B)$ is a family $(A \xrightarrow{f_s} B)_{s \in S}$ of maps in $\mathcal{A}$, or equivalently a map $\sum_{s \in S} A \longrightarrow B$. So the left adjoint is $S \longmapsto \sum_{s \in S} A$. We usually write $\sum_{s \in S} A$ as $S \times A$ and call it a **copower** of A; compare powers (page 70). To explain the notation, if 2 is a two-element set then $2 \times A = A + A$, and similarly for other numbers. Also, if $\mathcal{A} = \mathbf{Set}$ then the copower $S \times A$ is the same as the product $S \times A$.

2.(a) Bookwork.

(b) U does not preserve initial objects.

(c) I does not preserve the sum $1 + 1$.

C does not preserve the equalizer of the two distinct functors $\mathbf{1} \rightrightarrows \mathbf{2}$, where $\mathbf{2} = (\bullet \longrightarrow \bullet)$.

3. Bookwork.

4.(a) Straight application of definitions of pullback and monic.

(b) Just need to confirm that if $X_1 \xrightarrow{m_1} A$ and $X_2 \xrightarrow{m_2} A$ are monics representing same subobject of A then the monics $X_1' \xrightarrow{m_1'} A$ and $X_2' \xrightarrow{m_2'} A$ obtained by pulling back along f represent same subobject of A'. Can do this directly or prove a more general—and morally obvious—statement about isomorphic cones having isomorphic limits.

(c) Just need to check that Sub preserves identities (easy) and composition (direct from hint in question).

(d) Saw in Sheet 9, q.5 that in **Set**, subobjects are subsets. Saw in 4.1.16 that inverse images of subsets correspond to pullbacks of inclusions. From this, deduce that $\mathrm{Sub} \cong \mathcal{P}$, where $\mathcal{P}$ is as in 3.1.10(b). But saw there that $\mathcal{P} \cong H_2$, so $\mathrm{Sub} \cong H_2$.